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Convergence Analysis of Block Implicit One-Step Methods for Solving Differential/Algebraic Equations

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Convergence Analysis of Block Implicit One-Step Methods for Solving Differential/Algebraic Equations

__Abstract__: We will prove that numerical approximants, generated by fixed-stepsize, fixed-formula (FSFF) block implicit one-step methods, applied to smooth, nonlinear, decoupled systems of differential/algebraic equations converge to the true analytic solution.

__Key Words:__ fixed-stepsize, fixed-formula block implicit one-step methods, Banach space, Fréchet derivative, Kronecker product notation.

Introduction:

In this paper, we will consider some theoretical properties of fixed-stepsize, fixed-formula block-implicit one step methods that underly our numerical implementation in [5]. Of central importance in this theoretical discussion is the question of convergence. We will determine whether the approximants generated by our numerical algorithm approach the corresponding analytical solution as the stepsize h approaches zero - and, if so, what order in h this convergence follows.

The first section of our paper will contain a derivation of our FSFF method and some theoretical tools necessary to prove the convergence of these methods. The second section of our paper will contain some theoretical tools necessary to prove the convergence of FSFF block methods. In section three we will prove that our FSFF block implicit methods are consistent. In section four we will prove that our FSFF block implicit methods are stable. Finally, in section five we will use the fact that consistency and stability imply convergence.

1. Derivation of FSFF One-Step Block Implicit Methods

Suppose we consider an initial value problem of the form

$$\underline{y}'(x) = \underline{f}(x, \underline{y}(x)), \tag{1.1}$$

subject to the initial conditions

$$\underline{y}(x_0) = \underline{y}_0$$

such that

$$x \in [x_0, x_{fin}],$$

$$\underline{y}: [x_0, x_{fin}] \rightarrow \mathbf{R}^n,$$

and

$$\underline{f}: [x_0, x_{fin}] \times \mathbf{R}^n \rightarrow \mathbf{R}^n$$

is ($\mathcal{C}^0$, Lip) on the region $[x_0, x_{fin}] \times \mathbf{R}^n$.

We can think of the main idea behind fixed-stepsize, fixed-formula (FSFF) block-implicit one-step methods for solving (1.1), as going from x_k to x_{k+r} by approximating the integral in

$$\underline{y}(x_{k+r}) - \underline{y}(x_k) - \int_{x_k}^{x_{k+r}} \underline{f}(x, \underline{y}(x))\, dx = \underline{0}. \tag{1.2}$$

We are particularly interested in the class of FSFF block implicit one-step methods analyzed by H. A. Watts in [7]. Watts considers the grid points

$$\{x_j \mid x_j := x_0 + jh,\ rh \in (0, x_{fin} - x_0];\ j = 0, 1, 2, \ldots\}, \tag{1.3}$$

and generates the sequence $\{\underline{y}_j\}$, such that $\underline{y}_j$ approximates $\underline{y}(x_j)$ (the analytical solution to (1.1) at the point x_j). Watts defines a FSFF block implicit one-step method as a procedure which, when applied to a single equation of the form (1.1), yields r additional values $y_{k+1}, y_{k+2}, \ldots, y_{k+r}$ simultaneously at each stage of the application of the method where $k = mr$, for $m = 0, 1, 2, \ldots$.

A FSFF block implicit one-step formula applied to a single explicit ordinary differential equation of the form (1.1) takes on the following form in [7]:

$$\sum_{j=1}^{r} \mathfrak{a}_{ij} y_{k+j} = \mathfrak{e}_i y_k + h \mathfrak{d}_i z_k + h \sum_{j=1}^{r} \mathfrak{b}_{ij} z_{k+j}, \qquad (1.4)$$

such that

$\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{e}_i, \mathfrak{d}_i \in \mathbf{R}$

and $1 \le i,j \le r$.

Suppose for notational convenience we let

$\mathfrak{A} := [\mathfrak{a}_{ij}]$,

$\mathfrak{B} := [\mathfrak{b}_{ij}]$,

$\underline{\mathfrak{e}} = [\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_r]^T$,

$\underline{\mathfrak{d}} = [\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_r]^T$,

$\mathbf{y}_m = [y_{k+1}, y_{k+2}, \ldots, y_{k+r}]^T$,

$\underline{z} := \underline{y}'$,

$\underline{z}_k := \underline{y}'_k$,

$\mathbf{z}_m = [z_{k+1}, z_{k+2}, \ldots, z_{k+r}]^T$,

$1 \le i,j \le r$,

$$k = mr,$$

and

$$m = 0, 1, 2, \ldots .$$

Therefore, equations (1.4) can be concisely written as

$$\mathcal{A}\mathbf{y}_m = h\mathbf{B}\underline{\mathbf{z}}_m + \underline{\boldsymbol{e}}y_k + h\underline{\mathbf{d}}z_k \tag{1.5}$$

Assuming that $\mathcal{A}^{-1}$ exists, we will further simplify system (1.5) to obtain

$$\mathbf{y}_m = hB\underline{\mathbf{z}}_m + \underline{e}y_k + h\underline{d}z_k, \tag{1.6}$$

such that,

$$B := \mathcal{A}^{-1}\mathbf{B},$$

$$\underline{d} := \mathcal{A}^{-1}\underline{\mathbf{d}}$$

and

$$\underline{e} := \mathcal{A}^{-1}\boldsymbol{e} := [1, 1, \ldots, 1]^T.$$

Utilizing the Kronecker product notation, FSFF block implicit one-step methods may be applied to a nonlinear, decoupled system of differential/algebraic equations of the form

$$\underline{F}(x,\underline{y}(x),\underline{z}_d(x)) := \begin{bmatrix} \underline{F}_d(x,\underline{y}_d(x),\underline{y}_a(x),\underline{z}_d(x)) \\ \underline{F}_a(x,\underline{y}_d(x),\underline{y}_a(x)) \end{bmatrix} = \underline{0}, \tag{1.7}$$

subject to the initial conditions

$$\underline{y}(x_0) = \underline{y}_0,$$

such that

$$x \in [x_0, x_{fin}].$$

$\underline{F}:[x_0,x_{fin}] \times \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$

$\underline{F}_d:[x_0,x_{fin}] \times \mathbf{R}^{nd} \times \mathbf{R}^{na} \times \mathbf{R}^{nd} \rightarrow \mathbf{R}^{nd}$

$\underline{F}_a: [x_0,x_{fin}] \times \mathbf{R}^{nd} \times \mathbf{R}^{na} \rightarrow \mathbf{R}^{na}$

$\underline{y}_d:[x_0,x_{fin}] \rightarrow \mathbf{R}^{nd}$

$\underline{y}_a:[x_0,x_{fin}] \rightarrow \mathbf{R}^{na}$

$\underline{y}(x) := [\underline{y}_d^T(x),\underline{y}_a^T(x)]^T,$

$\underline{z} := \underline{y}'$

$n := nd + na,$

and the Jacobian matrix $[\partial\underline{F}_d/\partial\underline{z}_d]$ is of maximal rank on $[x_0,x_{fin}]$.

If $\underline{F}$ is twice continuously differentiable, then it can be proved, via generalized Picard-Lindelof theory [5], that the solutions to systems of the form (1.7) exist and are unique.

For notational convenience, let us define the following nr-dimensional arrays $\underline{\underline{y}}_m$, $\underline{\underline{z}}_m$, $\underline{\underline{y}}_0$, and $\underline{\underline{z}}_0$, to be

$\underline{\underline{y}}_m := [\underline{y}_{k+1}^T, \underline{y}_{k+2}^T, \ldots, \underline{y}_{k+r}^T]^T,$

$\underline{\underline{z}}_m := \underline{\underline{y}}'_m := [\underline{y}'_{k+1}{}^T, \underline{y}'_{k+2}{}^T, \ldots, \underline{y}'_{k+r}{}^T]^T,$

$\underline{\underline{y}}_0 := [\underline{y}_0^T, \underline{y}_0^T, \ldots, \underline{y}_0^T]^T,$

$\underline{\underline{z}}_0 := \underline{\underline{y}}'_0 := [\underline{z}_0^T, \underline{z}_0^T, \ldots, \underline{z}_0^T]^T,$

$k = mr,$

and

$m = 1, 2, 3, \ldots .$

Let us utilize the Kronecker product notation to define the following matrices:

$P := \mathcal{O}_r \mathbf{X} I_n,$

$Q := B \mathbf{X} I_n,$

$R := D_r \mathbf{X} I_n,$

such that

$$\mathcal{O}_r := \begin{bmatrix} 0 \dots 0 \; 1 \\ 0 \dots 0 \; 1 \\ \dots \\ 0 \dots 0 \; 1 \end{bmatrix},$$

$$D_r := \begin{bmatrix} 0 & \dots & 0 & d_1 \\ 0 & \dots & 0 & d_2 \\ & \dots & & \\ 0 & \dots & 0 & d_r \end{bmatrix},$$

$k = mr,$

$m = 0, 1, 2, \dots,$

$\mathcal{O}_r, D_r \in R^{r\times r},$

and I_n is the $n\times n$ identity matrix.

To advance our numerical method from $x_k := x_{mr}$ to $x_{k+r} := x_{(m+1)r}$, we will require that $\underline{y}_m$, associated with the interval $[x_k, x_{k+r}]$, satisfy the equations

$$\underline{F}_d(x_{k+i}, \underline{y}_{d,k+i}, \underline{y}_{a,k+i}, \underline{z}_{d,k+i}) = \underline{\eta}_{d,k+i},$$
$$\underline{F}_a(x_{k+i}, \underline{y}_{d,k+i}, \underline{y}_{a,k+i}) = \underline{\eta}_{a,k+i}, \qquad (1.8)$$

for $1 \le i \le r$, and simultaneously satisfy the block discretization formulae

$$\underline{\mathbf{y}}_m = P\underline{\mathbf{y}}_{m-1} + h[Q\underline{\mathbf{z}}_m + R\underline{\mathbf{z}}_{m-1}]. \tag{1.9}$$

The second section of our paper will contain some theoretical tools necessary to prove the convergence of FSFF block methods. In section three we will prove that our FSFF block implicit methods are consistent. In section four we will prove that our FSFF block implicit methods are stable. Finally, in section five we will use the fact that consistency and stability imply convergence.

2. Theoretical Tools

The notation presented in the previous section will now be utilized to consider some theoretical properties of FSFF block methods. Of central importance is the question of convergence. We will determine whether approximants generated by our numerical algorithm approach the corresponding analytical solution as the stepsize h approaches zero - and, if so, what order in h this convergence follows. More specifically, we will prove convergence of FSFF block methods (1.4) when applied to smooth, nonlinear, decoupled systems of differential/algebraic equations of the form (1.7) via obtaining the following classical result:

"Consistency + Stability $\Rightarrow$ Convergence."

In [4] H. B. Keller presents a general abstract study of methods for approximating the solution of nonlinear problems in a Banach space setting. His basic result is as follows: *If a nonlinear problem has a unique local analytical solution, and its consistent approximating problem has a stable Lipschitz continuous linearization, then the approximating problem has a stable solution which is close to the analytical solution.*

Keller considers the problem

$$G(v) = 0, \qquad (2.1)$$

such that

$G: B_1 \rightarrow B_2,$

and B_i $(i = 1,2)$ are appropriate Banach spaces. On a family of Banach spaces, $\{B_1^h, B_2^h\}$, the family of approximating problems,

$$G_h(v_h) = 0, \tag{2.2}$$

where

$$G_h: B_1^h \rightarrow B_2^h,$$

and

$$0 < h \leq h_0,$$

are also considered in [4].

To relate problems (2.1) and (2.2), Keller requires that there exist a family of linear mappings $\{P_1^h, P_2^h\}$ where

$$P_i^h: B_i \rightarrow B_i^h, \tag{2.3}$$

and

$$\lim_{h \rightarrow 0} \|P_i^h v\| = \|v\|, \tag{2.4}$$

for all $v \in B_i$ $(i = 1,2)$. For notational convenience the following is defined in [4]:

$$[v]_h := P_i^h v, \tag{2.5}$$

such that $[v]_h \in B_i^h$ if $v \in B_i$ $(i = 1,2)$.

Before we define consistency, stability, and convergence, we will define a sphere about an arbitrary point in a Banach space B as follows:

Definition 2.1: The sphere of radius ρ, about $u \in B$, is defined as follows:

$$S_\rho(u) := \{v: v \in B, \|v-u\| \leq \rho, \rho > 0\}. \square \tag{2.6}$$

Now we are prepared to state the definitions of stability, consistency, and convergence, to be utilized in our analysis.

Definition 2.2 [4]: The family $\{G_h(\cdot)\}$ is ***stable*** for $u \in B_1$ if and only if for some $h_0 > 0$, $\rho > 0$, and some constant $M_\rho > 0$, independent of h,

$$\|v_h - w_h\| \le M_\rho \|G_h(v_h) - G_h(w_h)\|, \tag{2.7}$$

for all $h \in (0, h_0]$, and all $v_h, w_h \in S_\rho([u]_h)$.□

Definition 2.3 [4]: The family $\{G_h(\cdot)\}$ is ***consistent*** of order $p \ge 1$ with $G(\cdot)$ on $S_\rho(u)$ if and only if

$$\|\tau_h(v)\| := \|G_h([v]_h) - [G(v)]_h\| \le M(v)h^p, \tag{2.8}$$

for all $v \in S_\rho(u)$, and some bounded functional $M(v) \ge 0$ independent of h.□

Suppose u is a unique local analytical solution to system (2.1). If we substitute u for v in (2.8), we obtain the following:

$$\|\tau_h(u)\| := \|G_h([u]_h)\| \le M(u)h^p.\square \tag{2.8'}$$

The significance of definitions 2.2 and 2.3 derives from the following theorem from [4].

Theorem 2.1:

Let

$$G(u) = 0,$$

and

$$G_h(w_h) = 0,$$

for some $w_h \in S_\rho([u]_h)$, $\rho > 0$, and all $h \in (0,h_0]$. Let $\{G_h(\cdot)\}$ be stable for u and consistent of order $p \geq 1$ with $G(\cdot)$ at u.

Then

$$|||[u]_h - w_h|| \leq M_p M(u) h^p, \tag{2.9}$$

for $M_p > 0$ □

Utilizing theorem 2.1, we will formally define convergence, and the order of convergence as follows:

Definition 2.4: A sequence of numerical approximants $w_h \in S_\rho([u]_h)$ is said to ***converge*** to a unique local analytical solution u of (1.5) if

$$\lim_{h \to 0} |||[u]_h - w_h|| \to 0 \tag{2.10a}$$

for $\rho > 0$, and for all $h \in (0,h_0]$. The ***order of convergence*** of this sequence is **p** if there exists some bounded functional $M(u) \geq 0$, independent of h, such that

$$|||[u]_h - w_h|| \leq M(u) h^p . \square \tag{2.10b}$$

Let us recall the definition of a Fréchet derivative of a bounded mapping $H(\cdot)$, such that $H: B_1 \to B_2$.

Definition 2.5: If there exists a bounded, linear operator $L(v): B_1 \to B_2$, such that

$$\lim_{\|w\| \to 0} \frac{\|H(v+w) - H(v) - L(v)w\|}{\|w\|} = 0, \tag{2.11}$$

for all $w \in E_1$, then $L(v)$ is the ***Fréchet derivative*** of H at $v \in E_1$.□

We shall see that for our difference approximations, obtained via applying our FSFF block methods to systems of the form (1.7), consistency is determined by simple Taylor series expansions. Because the stability verification for general nonlinear problems of the form (1.7) is not a "standard" procedure, we will proceed as in [4]. In particular, we will reduce our stability study to that of a linearized problem, and then apply the following lemma and theorem from [4]:

Lemma 2.1:

Let the family of mappings $\{G_h(\cdot)\}$ have Fréchet derivatives $\{L_h(v_h)\}$ on some family of spheres $S_{\rho_0}(w_h)$ and satisfy for all $h \in (0,h_0]$:

(i) $\{L_h(w_h)\}$ have uniformly bounded inverses at the centers of the spheres: that is, for some $K_0 > 0$,

$$\|L_h^{-1}(w_h)\| \le K_0. \tag{2.12}$$

(ii) $\{L_h(v_h)\}$ are uniformly Lipschitz continuous on $S_{\rho_0}(w_h)$: that is, for some $K_L > 0$,

$$\|L_h(v_h) - L_h(\nu_h)\| \le K_L\|v_h - \nu_h\|, \tag{2.13}$$

for all $v_h, \nu_h \in S_{\rho_0}(w_h)$.

If $w_h = [u]_h$ for some $u \in E_1$, then the family $\{G_h(\cdot)\}$ is stable for u.□

The following theorem from [4] insures the existence of a family of solutions $\{w_h\}$ which approximate a local solution u of (2.1).

Theorem 2.2:

Let $v = u$ be a unique local solution of (2.1). Let the family $\{G_h(\cdot)\}$ be consistent of order $p \geq 1$ with $G(\cdot)$ at u. Let the hypothesis (i) and (ii) of Lemma 2.1 hold with $w_h := [u]_h$.

Then, for $\rho_0 > 0$ and $h_0 > 0$ sufficiently small, and for each $h \in (0, h_0]$, the problem

$$G_h(v_h) = 0 \tag{2.14}$$

has a unique solution $v_h = w_h \equiv S_{\rho_0}([u]_h)$. These solutions satisfy

$$\|[u]_h - w_h\| \leq M_{\rho_0} M(u) h^p, \tag{2.15}$$

for $M_{\rho_0} > 0$.□

The inequality in (2.15) is equivalent to the following.

"Order of Convergence = Order of Consistency."

Also, we have convergence if $p = \mathbf{p} \geq 1$

Now let us introduce the concepts of the local discretization (truncation) error, and the overall order of our FSFF block implicit one-step methods.

Definition 2.6. We will define $\underline{T}_m$ to be the ***local discretization error*** at the mth step: that is,

$$\underline{T}_m := [\underline{\mathbf{t}}_{k+1}{}^T, \underline{\mathbf{t}}_{k+2}{}^T, \dots, \underline{\mathbf{t}}_{k+r}{}^T]^T, \tag{2.16}$$

where

$$\underline{\mathbf{t}}_{k+i} := \underline{y}(x_k + ih) - \underline{y}(x_k) - h \sum_{j=1}^{r} b_{ij}\underline{z}(x_k + jh) - hd_i\underline{z}(x_k), \tag{2.17}$$

for

$1 \leq i \leq r$,

$k = mr$.

$m = 0, 1, 2, \ldots,$

$[\underline{y}^T, \underline{z}^T]^T = [\underline{y}_d^T, \underline{y}_s^T, \underline{z}_d^T, \underline{z}_s^T]^T,$

is a unique local analytical solution to the initial value problem (1.7) □

The authors in [2], [3], and [6] refer to $\underline{t}_{k+i}$ $(1 \le i \le r-1)$ as the ***internal local truncation errors.***

Definition 2.7: The ***overall order*** $\boldsymbol{\chi}_{order}$ of our FSFF block implicit one-step methods (1.9) is defined as

$$\boldsymbol{\chi}_{order} = \min\{\boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \ldots, \boldsymbol{\chi}_r\}, \tag{2.18}$$

such that

$$\underline{t}_{k+i} = \mathbf{c}_i h^{\boldsymbol{\chi}_i+1} + o(h^{\boldsymbol{\chi}_i+1}), \tag{2.19}$$

for

$\mathbf{c}_i \ne 0,$

$k = mr,$

$m = 0, 1, 2, \ldots,$

and

$1 \le i \le r.$ □

Watts shows in Appendix A of [7] that if r is even, then

$$\boldsymbol{\chi}_r = r + 2, \tag{2.20}$$

otherwise,

$$\boldsymbol{\chi}_i = r + 1, \tag{2.21}$$

for $1 \le i \le r$. Therefore, the overall order $\boldsymbol{\chi}_{order}$ of our FSFF block implicit one-step methods (1.9) is $r + 1$

3. FSFF Index One Consistency Analysis for Smooth Systems of Differential/Algebraic Equations

The mapping $G:B_1 \rightarrow B_2$, defined in (2.1), corresponds to

$$G\underline{w}(x) := \begin{bmatrix} \underline{y}_d(x) - \underline{z}_d(x) \\ \underline{F}_d(x,\underline{y}_d(x),\underline{y}_a(x),\underline{z}_d(x)) \\ \underline{F}_a(x,\underline{y}_d(x),\underline{y}_a(x)) \end{bmatrix} = \underline{0} \tag{3.1}$$

such that

$x \in [x_0,x_{fin}]$

$\underline{y}(x_0) = \underline{y}_0,$

$\underline{y} := [\underline{y}_d^T, \underline{y}_a^T]^T,$

$\underline{y}:[x_0,x_{fin}] \rightarrow \mathbf{R}^n,$

$\underline{y}_d:[x_0,x_{fin}] \rightarrow \mathbf{R}^{nd},$

$\underline{y}_a:[x_0,x_{fin}] \rightarrow \mathbf{R}^{na},$

$\underline{z}_d:[x_0,x_{fin}] \rightarrow \mathbf{R}^{nd},$

$nd + na = n,$

and

$\underline{w} := [\underline{y}^T, \underline{z}_d^T]^T.$

The Banach space B_1 consists of the norm to be defined in (3.2), and the set of functions

$\underline{w}:[x_0,x_{fin}] \rightarrow \mathbf{R}^{nd} \times \mathbf{R}^{na} \times \mathbf{R}^{nd},$

which are of class C^{r+1} on $[x_0,x_{fin}]$. We will denote this Banach space as

$\mathbf{C}^{r+1}([x_0, x_{fin}], \mathbf{R}^{nd} \times \mathbf{R}^{na} \times \mathbf{R}^{nd})$. The Banach space B_2 consists of the norm to be defined in (3.2), and the set of functions

$\underline{\nu} : [x_0, x_{fin}] \to \mathbf{R}^{nd} \times \mathbf{R}^{na} \times \mathbf{R}^{nd}$,

which are of class $\mathcal{C}^r$ on $[x_0, x_{fin}]$. We will denote this Banach space as $\mathbf{C}^r([x_0, x_{fin}], \mathbf{R}^{nd} \times \mathbf{R}^{na} \times \mathbf{R}^{nd})$. The elements of B_i are of the form

$$\underline{\nu} := [\underline{v}^T, \underline{\boldsymbol{\eta}}^T]^T$$

$$:= [\underline{v}_{[d]}^T, \underline{v}_{[a]}^T, \underline{\boldsymbol{\eta}}^T]^T,$$

such that

$\underline{v} = [\underline{v}_{[d]}^T, \underline{v}_{[a]}^T]^T$,

$\underline{v}_{[d]} := [v_1, \ldots, v_{nd}]^T$,

$\underline{v}_{[a]} := [v_{nd+1}, \ldots, v_n]^T$,

$\underline{v}_{[d]} : [x_0, x_{fin}] \to \mathbf{R}^{nd}$,

$\underline{v}_{[a]} : [x_0, x_{fin}] \to \mathbf{R}^{na}$,

$\underline{v} : [x_0, x_{fin}] \to \mathbf{R}^{n}$,

$\underline{\boldsymbol{\eta}} : [x_0, x_{fin}] \to \mathbf{R}^{nd}$,

and

$i = 1, 2$.

If $\underline{\nu} \in B_i$ $(i = 1, 2)$, then we will let the norm of $\underline{\nu}$ be defined as

$$\|\underline{\nu}\| := \sup\{\|\underline{v}\|_\infty, \|\underline{\boldsymbol{\eta}}\|_\infty\}$$

$$:= \sup\{\|\underline{v}_{[d]}\|_\infty, \|\underline{v}_{[a]}\|_\infty, \|\underline{\boldsymbol{\eta}}\|_\infty\}$$

$$:= \sup\{\{\sup_{1 \le j \le nd} |v_j(x)|, \sup_{nd+1 \le k \le n} |v_k(x)|, \sup_{1 \le l \le nd} |\boldsymbol{\eta}_l(x)|\} : x \in [x_0, x_{fin}]\} \qquad (3.2a)$$

If $\underline{y}$ is a solution to (1.5), then (3.2a) reduces to

$$\|\underline{y}\| := \sup\{\|\underline{y}\|_{\infty}, \|\underline{y}'_d\|_{\infty}\}$$

$$:= \sup\{\|\underline{y}_d\|_{\infty}, \|\underline{y}_a\|_{\infty}, \|\underline{y}'_d\|_{\infty}\}. \qquad (3.2b)$$

Let us consider the following family of grid points:

$$x_k := x_0 + kh, \qquad (3.3)$$

such that

$$h := (x_{fin} - x_0)/\mathcal{K},$$

$$\mathcal{K} = \mathcal{N}r,$$

and

$$0 \le m \le \mathcal{N}.$$

Because of the manner in which h is defined, we see that

$$x_{\mathcal{K}} := x_{fin}$$

For notational convenience let us define the following r·nd-dimensional arrays:

$$\underline{\mathbf{y}}_{[nd,m]} := [\underline{y}_{d,k+1}{}^T, \ldots, \underline{y}_{d,k+r}{}^T]^T,$$

$$\underline{\mathbf{y}}_{[nd,0]} := [\underline{y}_{d,0}{}^T, \ldots, \underline{y}_{d,0}{}^T]^T,$$

$$\underline{\mathbf{z}}_{[nd,m]} := [\underline{z}_{d,k+1}{}^T, \ldots, \underline{z}_{d,k+r}{}^T]^T,$$

$$\underline{\mathbf{z}}_{[nd,0]} := [\underline{z}_{d,0}{}^T, \ldots, \underline{z}_{d,0}{}^T]^T,$$

for

$$1 \le m \le \mathcal{N},$$

and

$$k = mr.$$

Also, let us utilize the Kronecker product notation to define the following matrices, which will be utilized to extract the equations corresponding to the differential variables from the difference equations defined in (1.9):

$P_{[nd]} := \mathcal{O}_r \mathbf{X} [I_{nd} \;\; 0_{nd,na}]$,

$Q_{[nd]} := B \mathbf{X} I_{nd}$,

and

$R_{[nd]} := D_r \mathbf{X} I_{nd}$,

where $\mathcal{O}_r$, B, and D_r are defined as in equation (1.8).

For each such grid of the form (3.3), we can define the following difference scheme:

$$[hC_{[nd]}]^{-1}\{\underline{\mathbf{y}}_{[nd,m]} - P_{[nd]}\underline{\mathbf{y}}_{m-1} - h\{Q_{[nd]}\underline{\mathbf{z}}_{[nd,m]} + R_{[nd]}\underline{\mathbf{z}}_{[nd,m-1]}\}\} = \underline{0}_{r\cdot nd}, \quad (3.4a)$$

$$\underline{F}_d(x_{k+i}, \underline{y}_{d,k+i}, \underline{y}_{a,k+i}, \underline{z}_{d,k+i}) = \underline{n}_{d,k+i}, \quad (3.4b)$$

$$\underline{F}_a(x_{k+i}, \underline{y}_{d,k+i}, \underline{y}_{a,k+i}) = \underline{n}_{a,k+i}, \quad (3.4c)$$

such that

$C_{[nd]} := \mathrm{diag}[\underline{e}_1{}^T(Q_{[nd]}+R_{[nd]})\underline{e}, \ldots, \underline{e}_r{}^T(Q_{[nd]}+R_{[nd]})\underline{e}]$,

$1 \le m \le \mathcal{M}$,

$k = mr$,

and

$1 \le i \le r$.

The Banach spaces $B_i{}^h$ (i = 1,2) consist of the norm to be defined in (3.5), and the $(n+nd)\mathcal{K}$-dimensional Euclidean space

$(\mathbf{R}^{nd} \times \mathbf{R}^{na} \times \mathbf{R}^{nd})^{\mathbf{K}}$. The elements of B_i^h are of the form

$$\underline{v}_h := [\underline{v}_1^T, \underline{\mathbf{r}}_1^T, \dots, \underline{v}_{\mathbf{K}}^T, \underline{\mathbf{r}}_{\mathbf{K}}^T]^T$$

$$:= [\underline{v}_{(1,d)}^T, \underline{v}_{(1,a)}^T, \underline{\mathbf{r}}_1^T, \dots, \underline{v}_{(\mathbf{K},d)}^T, \underline{v}_{(\mathbf{K},a)}^T, \underline{\mathbf{r}}_{\mathbf{K}}^T]^T,$$

such that

$\underline{v}_s := [\underline{v}_{(s,d)}^T, \underline{v}_{(s,a)}^T]^T,$

$\underline{v}_{(s,d)} := [v_{s,1}, \dots, v_{s,nd}]^T,$

$\underline{v}_{(s,a)} := [v_{s,nd+1}, \dots, v_{s,n}]^T,$

$\underline{v}_{(s,d)} \in \mathbf{R}^{nd},$

$\underline{v}_{(s,a)} \in \mathbf{R}^{na},$

$\underline{v}_s \in \mathbf{R}^{n},$

$\underline{\mathbf{r}}_s \in \mathbf{R}^{nd},$

$1 \le s \le \mathbf{K},$

and

$i = 1,2$

If $\underline{v}_h \in B_i^h$ $(i = 1,2)$, then the norm of $\underline{v}_h$ will be defined as follows:

$$\|\underline{v}_h\| = \max\{\|\underline{v}_{(s,d)}\|_\infty, \|\underline{v}_{(s,a)}\|_\infty, \|\underline{\mathbf{r}}_s\|_\infty \; 1 \le s \le \mathbf{K}\}$$

$$:= \max\{\max_{1 \le j \le nd} |v_{s,j}|, \max_{nd+1 \le k \le n} |v_{s,k}|, \max_{1 \le l \le nd} |\mathbf{r}_{s,l}| : 1 \le s \le \mathbf{K}\}. \tag{3.5a}$$

If $\underline{v}$ is a solution to (2.1), then $\underline{\mathbf{r}}_s = \underline{v}'_{d,s}$ for $1 \le s \le \mathbf{K}$, and (3.5a) reduces to

$$\|\underline{v}_h\| := \max\{\|\underline{v}_s\|_\infty, \|\underline{v}'_{d,s}\|_\infty : 1 \le s \le \mathbf{K}\} \qquad (3.5$$

The mapping $G_h: B_1^h \to B_2^h$, defined in (2.1), and applied to

$$\underline{v}_h := [\underline{v}_1^T, \underline{z}_{d,1}^T, \dots, \underline{v}_{\mathbf{K}}^T, \underline{z}_{d,\mathbf{K}}^T]^T,$$

corresponds to the $(n + nd)\mathbf{K}$ discrete equations defined in (3.4).

The linear mappings $P_j^h : B_j \rightarrow B_j^h$ ($j = 1,2$), correspond to the mapping P^h, which is defined as follows.

$$[\underline{y}]_h := P^h\underline{y} := \{\underline{y}(x_s), \underline{\mathbf{r}}(x_s) : 1 \le s \le K\}, \qquad (3.6a)$$

such that

$\underline{y} \in B_j$,

$j = 1,2$,

and as will be shown,

$$\lim_{h \to 0} \|P^h\underline{y}\| = \|\underline{y}\| \qquad (3.7)$$

If $\underline{y}$ is a solution to (2.1), then $\underline{\mathbf{r}} := \underline{y}'_d$, and (3.6a) reduces to

$$[\underline{y}]_h := P^h\underline{y} := \{\underline{y}(x_s), \underline{y}'_d(x_s) : 1 \le s \le K\}. \qquad (3.6b)$$

Also, because we are working with smooth functions, we know that as the grid defined by (3.3) becomes denser, the maximum value in (3.4) will approach the supremum in (3.2). Thus the limit in (3.7) exists

3.1 Proof of Consistency

Now we are equipped to prove that our FSFF block implicit one-step methods (1.9), applied to smooth, nonlinear index one systems of the form (1.7), are consistent with $G(\cdot)$ at the solution $\underline{y} := [\underline{y}^T, \underline{z}_d^T]^T$ - as defined in definition 2.3. Using notation similar to that utilized in (2.8'), we obtain the following.

$$\tau_h(\underline{y}) = G_h([\underline{y}]_h)$$
$$= [\underline{\tau}_1(\underline{y})^T, \underline{\tau}_2(\underline{y})^T, \ldots, \underline{\tau}_M(\underline{y})^T]^T, \qquad (3.8)$$

such that

$$\underline{\tau}_m(\underline{y}) = \begin{bmatrix} [hC_{[nd]}]^{-1}\left\{\underline{Y}_{[nd,m]} - F_{[nd]}\underline{Y}_{m-1} - h\left\{D_{[nd]}\underline{Z}_{[nd,m]} + E_{[nd]}\underline{Z}_{[nd,m-1]}\right\}\right\} \\ \underline{F}_d(x_{k+1}, \underline{y}_{d,k+1}, \underline{y}_{a,k+1}, \underline{z}_{d,k+1}) \\ \underline{F}_a(x_{k+1}, \underline{y}_{d,k+1}, \underline{y}_{a,k+1}) \\ \vdots \\ \underline{F}_d(x_{k+r}, \underline{y}_{d,k+r}, \underline{y}_{a,k+r}, \underline{z}_{d,k+r}) \\ \underline{F}_a(x_{k+r}, \underline{y}_{d,k+r}, \underline{y}_{a,k+r}) \end{bmatrix}$$

via (2.16), where

$$\underline{T}_{[nd,m]} = [\underline{\tau}_{d,k+1}^T, \ldots, \underline{\tau}_{d,k+r}^T]^T$$
$$= [[\tau_{1,k+1}, \ldots, \tau_{nd,k+1}]^T, \ldots, [\tau_{1,k+r}, \ldots, \tau_{nd,k+r}]^T]^T,$$

$1 \le m \le M$,

$k = mr$

and

$1 \le i \le r$

Taking norms of both sides of (3.8) yields:

$$\|\tau_h(\underline{z})\|_\infty = \max\{\|[h\,C_{[nd]}]^{-1}\,\underline{T}_{[nd,m]}\|_\infty, \|\boldsymbol{\eta}_{[nd,m]}\|_\infty, \|\boldsymbol{\eta}_{[na,m]}\|_\infty : 1 \le m \le M\}$$
$$\le \max\{c\,\|[C_{[nd]}]^{-1}\|_\infty\, h^{r+1}, \max\{\|\boldsymbol{\eta}_{[nd,m]}\|_\infty, \|\boldsymbol{\eta}_{[na,m]}\|_\infty : 1 \le m \le M\}\},$$

(3.9)

such that

$\boldsymbol{\eta}_{[nd,m]} := [\underline{\eta}_{d,k+1}^T, \dots, \underline{\eta}_{d,k+r}^T]^T,$

$\boldsymbol{\eta}_{[na,m]} := [\underline{\eta}_{a,k+1}^T, \dots, \underline{\eta}_{a,k+r}^T]^T,$

$c := \max\{|c_i| : 1 \le i \le r\},$

where c_i $(1 \le i \le r)$ were defined in (2.19).

We can now summarize our FSFF index one consistency results as follows.

Theorem 3.1

Consider nonlinear, decoupled systems of differential/algebraic equations of the form (1.7), which satisfy the algebraic and local smoothness properties discussed. Let

$\underline{z} := [\underline{y}^T, \underline{z}_d^T]^T$

– a unique local analytical solution to (1.7) – be of class C^{r+1} on $[x_0, x_{fin}]$. Suppose $\partial \underline{F}_a/\partial \underline{y}_a$ is of full rank on $[x_0, x_{fin}]$. We will arrange to solve the nonlinear root finding problem so accurately, such that there exists positive constants c_d and c_a, such that

$$\|\underline{\eta}_{[nd,m]}\|_\infty \le c_d h^{r+1},$$

$$\|\underline{\eta}_{[na,m]}\|_\infty \le c_a h^{r+1},$$

for $1 \le m \le \mathcal{M}$.

Then there exists an $M > 0$, independent of h, such that

$$\|\tau_h(\underline{x})\| \le M \cdot h^p,$$

where

$$h := (x_{fin} - x_0)/\mathcal{K},$$

and

$$p = r+1,$$

that is, our PSPP block implicit one-step methods (1.9), applied to index one systems of the form (1.7) are consistent.□

Proof of Theorem 3.1

From (3.9) we have

$$\begin{aligned}\|\tau_h(\underline{x})\| &\le \max\{c\cdot\|[C_{[nd]}]^{-1}\|_\infty h^{r+1}, \|\underline{\eta}_{[nd,m]}\|_\infty, \|\underline{\eta}_{[na,m]}\|_\infty : 1 \le m \le \mathcal{M}\}\\ &\le \max\{c\cdot\|[C_{[nd]}]^{-1}\|_\infty h^{r+1}, c_d h^{r+1}, c_a h^{r+1}\}\\ &= \max\{c\cdot\|[C_{[nd]}]^{-1}\|_\infty, c_d, c_a\}\, h^{r+1}\\ &=: M\, h^{r+1}.\end{aligned}$$

□

4. FSFF Index One Stability Analysis for Smooth Systems of Differential/Algebraic Equations

Now we will prove stability of our FSFF block implicit one-step methods (1.9) applied to nonlinear, decoupled index one systems of differential/algebraic equations of the form (3.1). Our stability analysis can be summarized as follows: The family of mappings $G_h : B_1{}^h \to B_2{}^h$, defined in equations (3.4), will be considered. It will be shown that $L_h(\{\underline{x}\}_h)$, the Fréchet derivative of G_h at $\{\underline{x}\}_h$, exists on $S_{\rho_0}(\{\underline{x}\}_h)$, where $\underline{x} := [\underline{y}^T, \underline{z}_d{}^T]^T$ is a unique local analytical solution to (3.1). Then it will be shown that $L_h(\{\underline{x}\}_h)$ satisfies (2.12) and (2.13) – the hypothesis of Lemma 2.1.

To verify (2.12), it will be shown that if $\{\underline{\mathcal{U}}\}_M$ is the unique solution to

$$L_h(\{\underline{x}\}_h)\,\{\underline{\mathcal{U}}\}_M = \{\boldsymbol{\mathcal{V}}\}_M \tag{4.1}$$

(where $\{\underline{\mathcal{U}}\}_M$ and $\{\boldsymbol{\mathcal{V}}\}_M$ will be defined shortly), and if there exists a constant $M > 0$, such that

$$\|\{\underline{\mathcal{U}}\}_M\| \le M\|\{\boldsymbol{\mathcal{V}}\}_M\|, \tag{4.2}$$

then

$$\|L_h(\{\underline{x}\}_h)^{-1}\| \le M; \tag{4.3}$$

that is, $L_h(\{\underline{x}\}_h)^{-1}$ is uniformly bounded at the center of the sphere $S_{\rho_0}(\{\underline{x}\}_h)$ for all $h \in (0, \boldsymbol{h}]$.

The second hypothesis of Lemma 2.1 – (2.13) – will be shown to be satisfied if it is assumed that the function $\underline{F}$ has continuous, and bounded second partial derivatives with respect to $\underline{y}$ and $\underline{z}_d$. With all this in mind, We are now equipped to prove stability of the FSFF block implicit one-step methods (1.5), applied to nonlinear, decoupled index one systems of differential/algebraic equations of the form (3.1).

4.1 Derivation of the Fréchet Derivative

Suppose we consider $\underline{w}_h = S_{h0}(\{\underline{y}\}_h)$, such that

$\underline{w}_h := [\underline{w}_1^T, \underline{\mathbf{w}}'_1{}^T, \ldots, \underline{w}_{\mathcal{K}}^T, \underline{\mathbf{w}}'_{\mathcal{K}}{}^T]^T,$

$\underline{w}_s \approx \underline{y}(t_s),$

$\underline{\mathbf{w}}'_s \approx \underline{z}_d(t_s),$

and

$1 \le s \le \mathcal{K}$

The vector $\underline{w}_h$ satisfies the following difference scheme:

$$[hC_{[nd]}]^{-1}[\underline{\boldsymbol{\omega}}_{[nd,m]} - hD_{[nd]}\underline{\boldsymbol{\omega}}'_{[nd,m]} - P_{[nd]}\underline{\boldsymbol{\omega}}_{m-1} - hR_{[nd]}\underline{\boldsymbol{\omega}}'_{[nd,m-1]}] + \underline{\boldsymbol{\Gamma}}_{[nd,m]} = \underline{0}_{r \cdot nd},$$

$$\underline{F}_d(t_{k+i}, \underline{w}_{k+i}, \underline{\mathbf{w}}'_{k+i}) = \underline{\eta}_{\underline{w},d,k+i},$$

$$\underline{F}_a(t_{k+i}, \underline{w}_{k+i}) = \underline{\eta}_{\underline{w},a,k+i}, \qquad (4.4)$$

where

$\underline{\boldsymbol{\omega}}_m := [\underline{w}_{k+1}^T, \underline{w}_{k+2}^T, \cdots, \underline{w}_{k+r}^T]^T,$

$\underline{\boldsymbol{\omega}}_{[nd,m]} := [\underline{w}_{d,k+1}^T, \cdots, \underline{w}_{d,k+r}^T]^T,$

$\underline{\boldsymbol{\omega}}'_{[nd,m]} := [\underline{\mathbf{w}}'_{k+1}{}^T, \cdots, \underline{\mathbf{w}}'_{k+r}{}^T]^T,$

$\underline{\boldsymbol{\Gamma}}_{[nd,m]} := [\underline{\Gamma}_{d,k+1}^T, \cdots, \underline{\Gamma}_{d,k+r}^T]^T,$

$\underline{\boldsymbol{\eta}}_{\underline{w},m} := [\underline{\eta}_{\underline{w},k+1}^T, \cdots, \underline{\eta}_{\underline{w},k+r}^T]^T,$

$1 \le m \le \mathcal{M},$

$k = mr,$

and

$1 \le i \le r$

There is a residual term, $\underline{\mathbf{r}}_{[nd,m]}$, in (4.4) because $\underline{\mathcal{W}}_h$ does not satisfy the difference scheme (3.4) exactly.

Subtracting system (4.4) from system (3.4) yields the following:

$$[hC_{[nd]}]^{-1}\{\underline{\mathbf{u}}_{[nd,m]} - hQ_{[nd]}\underline{\mathbf{u}}'_{[nd,m]} - P_{[nd]}\underline{\mathbf{u}}_{m-1} - hR_{[nd]}\underline{\mathbf{u}}'_{[nd,m-1]}\} = \underline{\mathbf{r}}_{[nd,m]},$$

$$\underline{F}_d(x_{k+i}, \underline{y}_{k+i}, \underline{z}_{d,k+i}) - \underline{F}_d(x_{k+i}, \underline{w}_{k+i}, \underline{\mathbf{w}}_{k+i}) = \Delta\underline{r}_{\underline{w},d,k+i},$$

$$\underline{F}_a(x_{k+i}, \underline{y}_{k+i}) - \underline{F}_a(x_{k+i}, \underline{w}_{k+i}) = \Delta\underline{r}_{\underline{w},a,k+i}, \qquad (4.5)$$

such that

$\underline{\mathbf{u}}_m := \underline{\mathbf{y}}_m - \underline{\mathbf{w}}_m,$

$\underline{\mathbf{u}}_{[nd,m]} := \underline{\mathbf{y}}_{[nd,m]} - \underline{\mathbf{w}}_{[nd,m]},$

$\underline{\mathbf{u}}_{[na,m]} := \underline{\mathbf{y}}_{[na,m]} - \underline{\mathbf{w}}_{[na,m]},$

$\underline{\mathbf{u}}'_{[nd,m]} := \underline{\mathbf{z}}_{[nd,m]} - \underline{\mathbf{w}}'_{[nd,m]},$

$\underline{\mathcal{U}}_m := [\underline{\mathbf{u}}_m^T, \underline{\mathbf{u}}'^T_{[nd,m]}]^T,$

$[\underline{\mathcal{U}}]_M := [\underline{\mathcal{U}}_1^T, \ldots, \underline{\mathcal{U}}_M^T]^T,$

$\Delta\underline{\mathbf{r}}_{\underline{w},m} := [\Delta\underline{r}_{\underline{w},k+1}^T, \ldots, \Delta\underline{r}_{\underline{w},k+r}^T]^T,$

$\Delta\underline{r}_{\underline{w},k+i} := \underline{r}_{k+i} - \underline{r}_{\underline{w},k+i},$

$1 \le m \le M,$

$k = mr$

and

$1 \le i \le r$

The Fréchet derivative, $L_h([\underline{\mathcal{U}}]_h)$, of the mapping $G_h([\underline{\mathcal{U}}]_h)$ is obtained by linearizing (4.5), that is,

$$[hC_{[nd]}]^{-1}\{\underline{U}_{[nd,m]} - hQ_{[nd]}\underline{U}'_{[nd,m]} - P_{[nd]}\underline{U}_{m-1} - hR_{[nd]}\underline{U}'_{[nd,m-1]}\} = \underline{\Gamma}_{[nd,m]},$$

$$[\partial \underline{F}_d(\underline{\mathbf{P}}_{k+i})/\partial \underline{y}_d]\,\underline{u}_{d,k+i} + [\partial \underline{F}_d(\underline{\mathbf{P}}_{k+i})/\partial \underline{y}_a]\,\underline{u}_{a,k+i} + [\partial \underline{F}_d(\underline{\mathbf{P}}_{k+i})/\partial \underline{z}_d]\,\underline{u}'_{d,k+i} = \Delta \underline{r}_{\underline{w},d,k+i},$$

$$[\partial \underline{F}_a(\underline{\mathbf{P}}_{k+i})/\partial \underline{y}_d]\,\underline{u}_{d,k+i} + [\partial \underline{F}_a(\underline{\mathbf{P}}_{k+i})/\partial \underline{y}_a]\,\underline{u}_{a,k+i} = \Delta \underline{r}_{\underline{w},a,k+i}, \tag{4.6}$$

such that

$\underline{\mathbf{P}}_{k+i} := (t_{k+i}, \underline{y}_{k+i}, \underline{z}_{d,k+i}),$

$1 \le m \le M,$

$k = mr,$

and

$1 \le i \le r.$

Let us utilize the right hand side of (4.6) to make the following definition.

Definition 4.1: The *Fréchet derivative*, $L_h([\underline{y}]_h)$, of $G_h([\underline{y}]_h)$ on $S_{\rho_0}([\underline{y}]_h)$ is defined as

$$L_h([\underline{y}]_h) := \mathrm{diag}[\mathcal{L}_1([\underline{y}]_h), \ldots, \mathcal{L}_M([\underline{y}]_h)], \tag{4.7}$$

such that

$$\mathcal{L}_m([\underline{y}]_h) := \mathcal{R}_{*m}\,\underline{\mathcal{U}}_m - \mathcal{L}_{*m}E^{-1}, \tag{4.8}$$

$$\mathcal{R}_{*m} = \left[\begin{array}{c|c} [hC_{[nd]}]^{-1}\{I_r \otimes [I_{nd}\ 0_{nd,na}]\} & [hC_{[nd]}]^{-1}[-hQ_{[nd]}] \\ \hdashline \mathcal{D}_m & \mathcal{D}'_m \end{array}\right]$$

$$\mathbf{L}_{+m} := \left[\begin{array}{c|c} \left[hC_{[nd]}\right]^{-1} P_{[nd]} & \left[hC_{[nd]}\right]^{-1}\left[-hR_{[nd]}\right] \\ \text{-------} & \text{-------} \\ 0_{rn} & 0_{rn,rnd} \end{array}\right],$$

$$\mathcal{D}_m := \operatorname{diag}[J^{k+1}, \ldots, J^{k+r}],$$

$$\mathcal{D}'_m := \operatorname{diag}[\mathfrak{J}^{k+1}, \ldots, \mathfrak{J}^{k+r}],$$

$$J^{k+i} := \begin{bmatrix} J_{dd}^{k+i} & J_{da}^{k+i} \\ J_{ad}^{k+i} & J_{aa}^{k+i} \end{bmatrix},$$

$$\mathfrak{J}^{k+i} := \begin{bmatrix} \mathfrak{J}_{dd}^{k+i} \\ 0_{na,nd} \end{bmatrix},$$

for

$1 \le m \le M,$

$k = mr,$

and

$1 \le i \le r.$

Also, recall that E^{-1} is the backward shift operator: that is,

$E^{-1}\underline{\mathcal{U}}_m = \underline{\mathcal{U}}_{m-1},$

for $1 \le m \le M$, and $\underline{\mathcal{U}}_0 = \underline{0}$ □

We observe that if we define

$\{\underline{\boldsymbol{y}}\}_M := [\,\boldsymbol{y}_1^T, \ldots, \boldsymbol{y}_M^T]^T$

where

$$\boldsymbol{X}_m := [\underline{\Gamma}_{(nd,m)}{}^T, \Delta\underline{\eta}_{\underline{bw},m}{}^T]^T$$

and

$$1 \le m \le M,$$

then the equations in (4.6) can be concisely rewritten as system (4.1)

4.2. Proof of Fundamental Inequality

For notational convenience, we will refer to the inequality defined in (4.2) as our ***Fundamental Inequality*** (F.I.). By proving that our F.I. is true, we will also have satisfied (2.12) of lemma 2.1. Utlizing (4.8), we see that system (4.6) can be rewritten in block matrix notation as follows

$$\mathcal{R}_m \underline{\mathcal{U}}_m = \underline{\mathcal{S}}_m, \tag{4.9}$$

such that

$$\mathcal{R}_m := \left[\begin{array}{c|c} I_r \mathbf{X} \begin{bmatrix} I_{nd} & 0_{nd,ns} \end{bmatrix} & -hQ_{[nd]} \\ \hline \Omega_m & \Omega'_m \end{array}\right],$$

$$\underline{\mathcal{S}}_m := L_m \underline{\mathcal{U}}_{m-1} + \mathcal{Y}_{m,h}$$

$$:= \left[\begin{array}{c} P_{[nd]} \underline{\mathbf{U}}_{m-1} + hR_{[nd]} \underline{\mathbf{U}}'_{[nd,m-1]} + hC_{[nd]} \underline{\mathbf{\Gamma}}_{[nd,m]} \\ \hline \Delta \boldsymbol{\eta}_m \end{array}\right],$$

$$\mathcal{L}_m := \left[\begin{array}{c|c} P_{[nd]} & hR_{[nd]} \\ \hline 0_{m} & 0_{m,nd} \end{array}\right],$$

and

$$\mathcal{Y}_{(m,h)} := [h\,\mathbf{C}_{[nd]} \underline{\mathbf{\Gamma}}_{[nd,m]}{}^T, \Delta \underline{\boldsymbol{\eta}}_m{}^T]^T,$$

for $1 \le m \le \mathcal{M}$

Now we will utilize Block Gaussian Elimination to solve system (4.9) for $\underline{\mathbf{u}}_{[nd,m]}$, $\underline{\mathbf{u}}_{[na,m]}$, and $\underline{\mathbf{u}}'_{[nd,m]}$. We will then utilize these vectors to obtain our F I (4.2). To obtain

$$\underline{\mathbf{u}}_{[nd+na,m]} := [\,\underline{\mathbf{u}}_{[nd,m]}{}^{T}, \underline{\mathbf{u}}_{[na,m]}{}^{T}]^{T},$$

we will solve the following matrix equation:

$$\begin{aligned} T_{m}\cdot\underline{\mathbf{u}}_{[nd+na,m]} = {} & \mathcal{T}_{m-1}\cdot\underline{\mathbf{u}}_{[nd+na,m-1]} \\ & + [h\cdot(C_{[nd]}\,\underline{\Gamma}_{[nd,m]} + Q_{[nd]}\cdot[\mathcal{M}_{[m,dd]}]^{-1}\cdot\Delta\mathbf{n}_{\underline{w},[nd,m]} \\ & + R_{[nd]}\cdot[\mathcal{M}_{[m-1,dd]}]^{-1}\cdot\Delta\mathbf{n}_{\underline{w},[nd,m-1]})^{T}, \Delta\mathbf{n}_{\underline{w},[na,m]}{}^{T}]^{T}, \end{aligned} \tag{4.10}$$

such that

$$T_{m} := \left[\begin{array}{c|c} I_{rnd}-hQ_{[nd]}\mathcal{M}_{[m,dd]} & -hQ_{[nd]}\mathcal{M}_{[m,da]} \\ ---- & ---- \\ \mathcal{M}_{[m,ad]} & \mathcal{M}_{[m,aa]} \end{array}\right],$$

$$\mathcal{T}_{m-1} := \left[\begin{array}{c|c} P_{[nd]}+hR_{[nd]}\mathcal{M}_{[m-1,dd]} & hR_{[nd]}\mathcal{M}_{[m-1,da]} \\ ---- & ---- \\ 0_{rna \times rnd} & 0_{rna} \end{array}\right].$$

$\mathcal{M}_{[m,cc]} = -\mathrm{diag}[[J_{dd}{}^{k+1}]^{-1}[J_{dd}{}^{k+1}], \ldots, [J_{dd}{}^{k+r}]^{-1}[J_{dd}{}^{k+r}]],$

$\mathcal{M}_{[m,da]} := -\mathrm{diag}[[J_{dd}{}^{k+1}]^{-1}[J_{da}{}^{k+1}], \ldots, [J_{dd}{}^{k+r}]^{-1}[J_{da}{}^{k+r}]],$

$\mathcal{M}_{[m,ad]} := \mathrm{diag}[J_{ad}{}^{k+1}, \ldots, J_{ad}{}^{k+r}],$

$\mathcal{M}_{[m,aa]} := \mathrm{diag}[J_{aa}{}^{k+1}, \ldots, J_{aa}{}^{k+r}],$

$\mathcal{M}_{[m,dd]} := \mathrm{diag}[J_{dd}{}^{k+1}, \ldots, J_{dd}{}^{k+r}],$

$\mathcal{M}_{[m,oo]} \equiv R^{r\cdot nd \times r\cdot nd},$

$\mathcal{M}_{[m,da]} \equiv R^{r\cdot nd \times r\cdot na}.$

$\mathcal{M}_{[m,ad]} \in \mathbb{R}^{r.na \times r.nd}$,

$\mathcal{M}_{[m,aa]} \in \mathbb{R}^{r.na \times r.na}$,

$\mathcal{M}_{[m,dd]} \in \mathbb{R}^{r.nd \times r.nd}$,

$\mathcal{P}_{[nd]} := \mathcal{O}_r \times I_{nd}$,

$\Delta\underline{\mathbf{n}}_{\underline{w},[nd,m]} := [\Delta\underline{\mathbf{n}}_{\underline{w},d,k+1}{}^T, \ldots, \Delta\underline{\mathbf{n}}_{\underline{w},d,k+r}{}^T]^T$,

and $1 \le m \le \mathcal{M}$. We will then utilize the solution to (4.10) to obtain $\underline{\mathbf{u}}'_{[nd,m]}$, that is,

$$\underline{\mathbf{u}}'_{[nd,m]} := \mathcal{M}_{[m,dd]}\,\underline{\mathbf{u}}_{[nd,m]} + \mathcal{M}_{[m,da]}\,\underline{\mathbf{u}}_{[na,m]} + [\mathcal{M}_{[m,dd]}]^{-1} \cdot \Delta\underline{\mathbf{n}}_{\underline{w},[nd,m]}, \tag{4.11}$$

for $1 \le m \le \mathcal{M}$.

Because $J_{aa}{}^{k+i}$ is nonsingular, system (4.10) can be solved for $\underline{\mathbf{u}}_{[na,m]}$ as follows:

$$\underline{\mathbf{u}}_{[na,m]} := [\mathcal{M}_{[m,aa]}]^{-1} \cdot \Delta\underline{\mathbf{n}}_{\underline{w},[na,m]} - [\mathcal{M}_{[m,aa]}]^{-1} \cdot \mathcal{M}_{[m,ad]}\,\underline{\mathbf{u}}_{[nd,m]}, \tag{4.12}$$

such that

$\Delta\underline{\mathbf{n}}_{\underline{w},[na,m]} := [\Delta\underline{\mathbf{n}}_{\underline{w},a,k+1}{}^T, \ldots, \Delta\underline{\mathbf{n}}_{\underline{w},a,k+r}{}^T]^T$,

for $1 \le i \le r$, $k = mr$, and $1 \le m \le \mathcal{M}$. Subsystem (4.12) is then substituted into (4.10) to obtain the following:

$$\begin{aligned} &\{I_{r\cdot nd} - h \cdot Q_{[nd]} \cdot \{\mathcal{M}_{[m,dd]} - \mathcal{M}_{[m,da]}\,[\mathcal{M}_{[m,aa]}]^{-1} \cdot \mathcal{M}_{[m,ad]}\}\}\,\underline{\mathbf{u}}_{[nd,m]} \\ &= \{\mathcal{P}_{[nd]} + h\,E_{[nd]} \cdot \{\mathcal{M}_{[m-1,dd]} - \mathcal{M}_{[m-1,da]} \cdot [\mathcal{M}_{[m-1,aa]}]^{-1} \cdot \mathcal{M}_{[m-1,ad]}\}\}\,\underline{\mathbf{u}}_{[nd,m-1]} \\ &\qquad + h \cdot \underline{\mathbf{n}}_m(h), \end{aligned} \tag{4.13}$$

such that

$$\underline{\mathbf{n}}_m(h) := \{C_{[nd]}\cdot\underline{\Gamma}_{[nd,m]}$$
$$+ Q_{[nd]}\cdot[[h\cdot\mathcal{V}_{[m,dd]}]^{-1}\cdot\triangle\underline{\mathbf{n}}_{\underline{w},[nd,m]} + \mathcal{M}_{[m,da]}\cdot[\mathcal{M}_{[m,aa]}]^{-1}\cdot\triangle\underline{\mathbf{n}}_{\underline{w},[na,m]}]$$
$$+ R_{[nd]}\cdot[[h\cdot\mathcal{V}_{[m-1,dd]}]^{-1}\cdot\triangle\underline{\mathbf{n}}_{\underline{w},[nd,m-1]} + \mathcal{M}_{[m-1,da]}\cdot[\mathcal{M}_{[m-1,aa]}]^{-1}\cdot\triangle\underline{\mathbf{n}}_{\underline{w},[na,m-1]}]\},$$

for $1 \le m \le \mathcal{M}$.

Thus we obtain the following recurrence relationship:

$$\underline{\mathbf{u}}_{[nd,m]} := \mathcal{M}_m(h)\cdot\underline{\mathbf{u}}_{[nd,m-1]} + \underline{\mathbf{N}}_m(h), \tag{4.14}$$

such that

$$\mathcal{M}_m(h) := \{I_{r\cdot nd} - h\cdot Q_{[nd]}\cdot\{\mathcal{M}_{[m,dd]} - \mathcal{M}_{[m,da]}\cdot[\mathcal{M}_{[m,aa]}]^{-1}\cdot\mathcal{M}_{[m,ad]}\}\}^{-1}\cdot$$
$$\{P_{[nd]} + h\cdot R_{[nd]}\cdot\{\mathcal{M}_{[m-1,dd]} - \mathcal{M}_{[m-1,da]}\cdot[\mathcal{M}_{[m-1,aa]}]^{-1}\cdot\mathcal{M}_{[m-1,ad]}\}\},$$

$$\underline{\mathbf{N}}_m(h) := h\cdot\{I_{r\cdot nd} - h\cdot Q_{[nd]}\cdot\{\mathcal{M}_{[m,dd]} - \mathcal{M}_{[m,da]}\cdot[\mathcal{M}_{[m,aa]}]^{-1}\cdot\mathcal{M}_{[m,ad]}\}\}^{-1}\cdot\underline{\mathbf{n}}_m(h),$$

for $1 \le m \le \mathcal{M}$.

Taking norms of both sides of (4.14) yields,

$$\|\underline{\mathbf{u}}_{[nd,m]}\| \le \|\mathcal{M}_m(h)\|\cdot\|\underline{\mathbf{u}}_{[nd,m-1]}\| + \|\underline{\mathbf{N}}_m(h)\|, \tag{4.15}$$

for $1 \le m \le \mathcal{M}$. For notational convenience, we will let $\|\cdot\|$ stand for $\|\cdot\|_\infty$. Also, let us define a sequence of numbers, $\{V_m\}$, which satisfies the following difference equation.

$$V_m = \|\mathcal{M}_m(h)\|V_{m-1} + \|\underline{\mathbf{N}}_m(h)\|, \tag{4.16}$$

for $1 \le m \le \mathcal{M}$, and $V_0 \equiv 0$.

The following lemma is necessary to enable us to utilize the sequence $\{V_m\}$ in proving our F. I. (4.2):

Lemma 4.1

If the initial conditions for (4.14) are exactly satisfied – that is, $\underline{\mathbf{E}}_{[nd,0]} \equiv \underline{0}$, then

$$\|\underline{\mathbf{E}}_{[nd,m]}\| \le V_m, \tag{4.17}$$

for $1 \le m \le \mathcal{M}$.□

Proof of Lemma 4.1

The inequality (4.17) will be proven by induction as follows: Because $V_0 \equiv 0$, and $\underline{\mathbf{E}}_{[nd,0]} \equiv \underline{0}$, (4.17) is true for $m = 0$. Now suppose (4.17) is true for some $m < \mathcal{M}$. Then (4.15) and (4.16) imply

$$\begin{aligned} \|\underline{\mathbf{E}}_{[nd,m+1]}\| &\le \|\mathcal{M}_{m+1}(h)\| \cdot \|\underline{\mathbf{E}}_{[nd,m]}\| + \|\underline{\mathbf{N}}_{m+1}(h)\| \\ &\le \|\mathcal{M}_{m+1}(h)\| V_m + \|\underline{\mathbf{N}}_{m+1}(h)\| \\ &= V_{m+1}, \end{aligned} \tag{4.18}$$

for $1 \le m \le \mathcal{M}$.□

From the theory of finite difference equations, we know that (4.16) can be solved for V_m as follows:

$$\begin{aligned} V_m &= \left\{ \prod_{j=1}^{m} \|\mathcal{M}_j(h)\| \right\} \left\{ \sum_{s=1}^{m} \|\underline{\mathbf{N}}_s(h)\| \left[\prod_{l=1}^{s} \|\mathcal{M}_l(h)\| \right]^{-1} \right\} \\ &= \|\underline{\mathbf{N}}_1(h)\| \prod_{j=2}^{m} \|\mathcal{M}_j(h)\| + \|\underline{\mathbf{N}}_2(h)\| \prod_{j=3}^{m} \|\mathcal{M}_j(h)\| + \ldots + \|\underline{\mathbf{N}}_m(h)\| \end{aligned} \tag{4.19}$$

for $1 \le m \le \mathcal{M}$

Now we will prove that $\prod_{j=1}^{m}\|\mathcal{M}_j(h)\|$ and $\|\underline{\mathbf{N}}_m(h)\|$ are uniformly bounded for

$1 \le i \le m$, and $1 \le m \le \mathcal{M}$.

Lemma 4.2

Suppose there exist positive constants L_{dd}, L_{da}, L_{ad}, $\mathcal{L}_{aa}$, and $\mathbf{L}_{dd}$, such that

$\|J_{dd}^{k+i}\| \le L_{dd}$,

$\|J_{da}^{k+i}\| \le L_{da}$,

$\|J_{ad}^{k+i}\| \le L_{ad}$,

$\|(J_{aa}^{k+i})^{-1}\| \le \mathcal{L}_{aa}$,

and

$\|(J_{dd}^{k+i})^{-1}\| \le \mathbf{L}_{dd}$,

for $1 \le i \le r$, and $k = mr$. Finally, suppose that $\boldsymbol{\hbar}$ is small enough, such that

$h\mathcal{R} < 1$,

where

$h \in (0, \boldsymbol{\hbar}]$,

and

$\mathcal{R} := \|B\|\,\mathbf{L}_{dd}\,(L_{dd} + L_{da}\mathcal{L}_{aa}L_{ad})$.

Then there exist positive constants M_{stab} and N_{stab}, such that

$$\prod_{j=1}^{m}\|\mathcal{M}_j(h)\| \le M_{stab}, \tag{4.20}$$

and

$$\|\underline{\mathbf{N}}_m(h)\| \le h \cdot M_{stab} \|(\boldsymbol{\mathcal{D}})_{\mathcal{M}}\|, \qquad (4.21)$$

for $1 \le i \le m$, and $1 \le m \le \mathcal{M}$.□

Proof of Lemma 4.2

First we will tackle the inequality in (4.20): that is, let us obtain M_{stab}. From (4.14) we see that

$$\|\mathcal{M}_m(h)\| \le \alpha_1 \cdot \alpha_2, \qquad (4.22)$$

where

$$\alpha_1 := \|\{I_{r \cdot nd} - h \cdot C_{[nd]} \{\mathcal{M}_{[m,dd]} - \mathcal{M}_{[m,da]}[\mathcal{M}_{[m,aa]}]^{-1} \mathcal{M}_{[m,ad]}\}\}^{-1}\|,$$

and

$$\alpha_2 := \|\mathcal{P}_{[nd]} + h \cdot R_{[nd]} \cdot \{\mathcal{M}_{[m-1,dd]} - \mathcal{M}_{[m-1,da]}[\mathcal{M}_{[m-1,aa]}]^{-1} \mathcal{M}_{[m-1,ad]}\}\|,$$

for $1 \le m \le \mathcal{M}$.

Let us take a look at α_i ($i = 1,2$) to complete (4.22). We will now obtain a bound for α_1 via relating the norms of the various matrices in α_1 and to the Lipschitz constants in our hypothesis.

$$\|C_{[nd]}\| := \|C \mathbf{X} I_{nd}\| := \|C\|;$$

$$\|\mathcal{M}_{[m,dd]}\| := \|\mathrm{diag}[[J_{dd}^{k+1}]^{-1}[J_{dd}^{k+1}], \ldots, [J_{dd}^{k+r}]^{-1}[J_{dd}^{k+r}]]\|$$
$$\le \max\{\|[J_{dd}^{k+i}]^{-1}\| \cdot \|[J_{dd}^{k+i}]\| : 1 \le i \le r, k = mr\},$$
$$\le \mathbf{L}_{dd} \cdot L_{dd},$$

by the hypothesis;

$$\|\mathcal{M}_{[m,da]}\| := \|\mathrm{diag}[[J_{dd}^{k+1}]^{-1}[J_{da}^{k+1}], \ldots, [J_{dd}^{k+r}]^{-1}[J_{da}^{k+r}]]\|$$
$$\le \max\{\|[J_{dd}^{k+i}]^{-1}\| \, \|[J_{da}^{k+i}]\| : 1 \le i \le r, k = mr\},$$

$$\leq L_{dd} L_{da},$$

by the hypothesis,

$$\|[\mathcal{M}_{[m,aa]}]^{-1}\| := \|\mathrm{diag}[[U_{aa}^{k+1}]^{-1}, \ldots, [U_{aa}^{k+r}]^{-1}]\|$$

$$\leq \max\{\|[U_{aa}^{k+i}]^{-1}\|: 1 \leq i \leq r, k = mr\},$$

$$\leq L_{aa},$$

by the hypothesis; and

$$\|\mathcal{M}_{[m,ad]}\| := \|\mathrm{diag}[[U_{ad}^{k+1}], \ldots, [U_{ad}^{k+r}]]\|$$

$$\leq \max\{\|[U_{ad}^{k+i}]\|: 1 \leq i \leq r, k = mr\},$$

$$\leq L_{ad},$$

by the hypothesis, for $1 \leq m \leq \mathcal{N}$.

Now we will define the constant β_1, which will be utilized in our bound for α_1. This inequality is

$$h^2 \cdot \mathcal{R}^2 + h^3 \cdot \mathcal{R}^3 + \ldots \leq \beta_1 \cdot h \cdot \mathcal{R}, \tag{4.23}$$

where

$$\beta_1 := h \cdot \mathcal{R}/(1 - h \cdot \mathcal{R}).$$

Utilizing the individual bounds for the various norms in α_1, (4.23), and the hypothesis, we obtain:

$$\alpha_1 \leq (1 - \|h \cdot Q_{[nd]} \cdot (\mathcal{M}_{[m,dd]} - \mathcal{M}_{[m,da]} \cdot [\mathcal{M}_{[m,aa]}]^{-1} \cdot \mathcal{M}_{[m,ad]})\|)^{-1}$$

$$\leq (1 - h \cdot \|Q_{[nd]}\| \cdot (\|\mathcal{M}_{[m,dd]}\| + \|\mathcal{M}_{[m,da]}\| \cdot \|[\mathcal{M}_{[m,aa]}]^{-1}\| \cdot \|\mathcal{M}_{[m,ad]}\|))^{-1}$$

$$\leq (1 - h \cdot \mathcal{R})^{-1}$$

$$= 1 + h \cdot \mathcal{R} + h^2 \cdot \mathcal{R}^2 + \ldots$$

$$\leq 1 + h \cdot (1 + \beta_1) \cdot \mathcal{R}$$

$$\leq \exp(h \cdot (1 + \beta_1) \cdot \mathcal{R}), \tag{4.24}$$

for $h \in (0, \hbar]$.

$$\alpha_2 \le \|\boldsymbol{P}_{[nd]}\| +$$

$$h \cdot \|R_{[nd]}\| \cdot (\|\mathcal{M}_{[m-1,dd]}\| + \|\mathcal{M}_{[m-1,da]}\| \cdot \|[\mathcal{M}_{[m-1,aa]}]^{-1}\| \cdot \|\mathcal{M}_{[m-1,ad]}\|), \qquad (4.25)$$

for $1 \le m \le \mathcal{M}$. The following norms will be utilized to complete (4.25)

$$\|\boldsymbol{P}_{[nd]}\| := \|\mathcal{O}_r \times I_{nd}\| := \|\mathcal{O}_r\| = 1;$$

and

$$\|R_{[nd]}\| := \|D_r \times I_{nd}\| := \|D_r\| = \|\underline{d}\|.$$

Therefore (4.25) can be completed as follows:

$$\begin{aligned} \alpha_2 &\le 1 + h \cdot \|\underline{d}\| \cdot \mathcal{L}_{dd}(L_{dd} + L_{da}\mathcal{L}_{aa}L_{ad}) \\ &\le \exp\{h \cdot \|\underline{d}\| \cdot \mathcal{L}_{dd}(L_{dd} + L_{da}\mathcal{L}_{aa}L_{ad})\} \\ &= \exp\{h \cdot \|\underline{d}\| \cdot \mathcal{R}/\|B\|\}, \end{aligned} \qquad (4.26)$$

for $h \in (0, \hbar]$.

Therefore (4.22) can be completed as follows.

$$\begin{aligned} \|\mathcal{M}_m(h)\| &\le \exp\{h \cdot (1+\beta_1) \cdot \mathcal{R}\} \cdot \exp\{h \cdot \|\underline{d}\| \cdot \mathcal{R}/\|B\|\} \\ &= \exp\{h \cdot \mathcal{R}\,(1+\beta_1 + \|\underline{d}\|/\|B\|)\}, \end{aligned}$$

for $1 \le m \le \mathcal{M}$. Thus (4.20) can be completed as follows:

$$\begin{aligned} \prod_{j=1}^{m} \|\mathcal{M}_j(h)\| &\le \exp\{(m-i) \cdot h \cdot \mathcal{R}\,(1+\beta_1 + \|\underline{d}\|/\|B\|)\} \\ &\le \exp\{\mathcal{M}\,[(x_{fin} - x_0)/\mathcal{M}]\,\mathcal{R}\,(1+\beta_1 + \|\underline{d}\|/\|B\|)\} \\ &= \exp\{(x_{fin} - x_0)\,\mathcal{R}\,(1+\beta_1 + \|\underline{d}\|/\|B\|)\} \\ &=: M_{stab}, \end{aligned} \qquad (4.27)$$

for $1 \le i \le m$, and $1 \le m \le \mathcal{M}$.

Proof of Lemma 4.3

From (4.19) and lemma 4.2, we have

$$\begin{aligned}
V_m &\le h\cdot N_{stab}\cdot\|\{\boldsymbol{\mathcal{Y}}\}_{\mathcal{M}}\|\cdot M_{stab}+h\cdot N_{stab}\cdot\|\{\boldsymbol{\mathcal{Y}}\}_{\mathcal{M}}\|\cdot M_{stab}+\ldots+h\cdot N_{stab}\cdot\|\{\boldsymbol{\mathcal{Y}}\}_{\mathcal{M}}\| \\
&\le m\cdot h\cdot N_{stab}\cdot\max(1,M_{stab})\cdot\|\{\boldsymbol{\mathcal{Y}}\}_{\mathcal{M}}\| \\
&\le \mathcal{M}\cdot[(x_{fin}-x_0)/\mathcal{M}]\cdot N_{stab}\cdot\max(1,M_{stab})\cdot\|\{\boldsymbol{\mathcal{Y}}\}_{\mathcal{M}}\| \\
&\le (x_{fin}-x_0)\cdot N_{stab}\cdot\max(1,M_{stab})\cdot\|\{\boldsymbol{\mathcal{Y}}\}_{\mathcal{M}}\| \\
&=: C\cdot\|\{\boldsymbol{\mathcal{Y}}\}_{\mathcal{M}}\|,
\end{aligned} \tag{4.29}$$

for $1 \le m \le \mathcal{M}$.□

The following completes the proof of our F. I. (4.2) for index one systems.

Lemma 4.4

Suppose $\boldsymbol{\mathcal{N}}_{[m,aa]}$ is nonsingular for $1 \le m \le \mathcal{M}$: that is, J_{aa}^{k+i} is nonsingular for $1 \le i \le r$, and $k = mr$. Suppose the hypothesis of lemmas 4.1, 4.2, and 4.3 are true.

Then our F. I.

$$\|\{\underline{\mathcal{U}}\}_{\mathcal{M}}\| \le M\cdot\|\{\boldsymbol{\mathcal{Y}}\}_{\mathcal{M}}\|$$

is true for all $h \in (0,\boldsymbol{h}]$.□

Proof of Lemma 4.4

$$\begin{aligned}
\|\{\underline{\mathcal{U}}\}_{\mathcal{M}}\| &:= \|\underline{\mathcal{U}}_1^T,\ldots,\underline{\mathcal{U}}_{\mathcal{M}}^T\| \\
&\le \max\{\|\underline{\mathcal{U}}_m\|: 1 \le m \le \mathcal{M}\} \\
&:= \max\{\|[\underline{\mathbf{U}}_m^T,\underline{\mathbf{U}}'^T_{[nd,m]}]\|: 1 \le m \le \mathcal{M}\} \\
&\le \max\{\max\{\|\underline{\mathbf{U}}_{[nd,m]}\|, \|\underline{\mathbf{U}}_{[na,m]}\|, \|\underline{\mathbf{U}}'_{[nd,m]}\|\}: 1 \le m \le \mathcal{M}\}.
\end{aligned} \tag{4.30}$$

Now let us obtain N_{stab}. From (4.14) we have

$$\|\underline{\mathbf{N}}_m(h)\| \le h\cdot\alpha_1\cdot\|\underline{\mathbf{n}}_m(h)\|$$

$$\le h\alpha_1\{\|C_{[nd]}\|\cdot\|\underline{\Gamma}_{[nd,m]}\|+\|Q_{[nd]}\|\cdot[\,\|[\mathcal{M}_{[m,dd]}]^{-1}\|\cdot\|\Delta\underline{\mathbf{n}}_{\underline{w},[nd,m]}\|+$$
$$\|\mathcal{M}_{[m,da]}\|\,\|[\mathcal{M}_{[m,aa]}]^{-1}\|\cdot\|\Delta\underline{\mathbf{n}}_{\underline{w},[na,m]}\|\,]+\|R_{[nd]}\|\cdot[\,\|[\mathcal{M}_{[m-1,dd]}]^{-1}\|\cdot\|\Delta\underline{\mathbf{n}}_{\underline{w},[nd,m-1]}\|$$
$$+\|\mathcal{M}_{[m-1,da]}\|\,\|[\mathcal{M}_{[m-1,aa]}]^{-1}\|\,\|\Delta\underline{\mathbf{n}}_{\underline{w},[na,m-1]}\|]\}$$

$$\le h\cdot\alpha_1\cdot\{\|C_{[nd]}\|\cdot\|\{\mathcal{Y}\}_{\mathcal{M}}\|+\|B\|\,[\mathbf{L}_{dd}\cdot\|\{\mathcal{Y}\}_{\mathcal{M}}\|+\mathbf{L}_{dd}\cdot L_{da}\cdot\mathcal{L}_{aa}\cdot\|\{\mathcal{Y}\}_{\mathcal{M}}\|]$$
$$+\|\underline{d}\|\cdot[\mathbf{L}_{dd}\cdot\|\{\mathcal{Y}\}_{\mathcal{M}}\|+\mathbf{L}_{dd}\cdot L_{da}\cdot\mathcal{L}_{aa}\cdot\|\{\mathcal{Y}\}_{\mathcal{M}}\|]\}$$

$$\le h\exp\{(x_{fin}-x_0)\cdot(1+\beta_1)\cdot\mathcal{R}\}\cdot\{\|\underline{C}_{[nd]}\|+$$
$$(\|B\|+\|\underline{d}\|)\cdot\mathbf{L}_{dd}\cdot[1+L_{da}\cdot\mathcal{L}_{aa}]\}\cdot\|\{\mathcal{Y}\}_{\mathcal{M}}\|$$

$$=: h\cdot N_{stab}\cdot\|\{\mathcal{Y}\}_{\mathcal{M}}\|,$$

for $1 \le m \le \mathcal{M}$, thus proving (4.21).□

Now we will obtain a bound for the expression defined in (4.19): that is,

Lemma 4.3

If the hypothesis of lemma 4.2 is true, then there exists a positive constant C, such that

$$V_m \le C\cdot\|\{\mathcal{Y}\}_{\mathcal{M}}\|, \tag{4.28}$$

for $1 \le m \le \mathcal{M}$.□

Recall from (4.17), and (4.28) that

$$
\begin{aligned}
\|\underline{\mathbf{u}}_{[nd,m]}\| &\le V_m \\
&\le C\cdot\|\{\boldsymbol{y}\}_{\mathcal{M}}\| \qquad (4.31)
\end{aligned}
$$

for $1 \le m \le \mathcal{M}$. Also, recall from (4.12) that

$$
\begin{aligned}
\|\underline{\mathbf{u}}_{[na,m]}\| &:= \|[\boldsymbol{M}_{[m,aa]}]^{-1}\cdot\Delta\boldsymbol{\eta}_{\underline{w},[na,m]} - [\boldsymbol{M}_{[m,aa]}]^{-1}\cdot\boldsymbol{M}_{[m,ad]}\cdot\underline{\mathbf{u}}_{[nd,m]}\| \\
&\le \|[\boldsymbol{M}_{[m,aa]}]^{-1}\|\cdot\{\|\Delta\boldsymbol{\eta}_{\underline{w},[na,m]}\| + \|\boldsymbol{M}_{[m,ad]}\|\cdot\|\underline{\mathbf{u}}_{[nd,m]}\|\} \\
&\le \mathcal{L}_{aa}\cdot\{1 + L_{ad}\cdot C\}\cdot\|\{\boldsymbol{y}\}_{\mathcal{M}}\|, \qquad (4.32)
\end{aligned}
$$

for $1 \le m \le \mathcal{M}$. Finally, recall from (4.11) that

$$
\begin{aligned}
\|\underline{\mathbf{u}}'_{[nd,m]}\| &:= \|\boldsymbol{M}_{[m,dd]}\cdot\underline{\mathbf{u}}_{[nd,m]} + \boldsymbol{M}_{[m,da]}\cdot\underline{\mathbf{u}}_{[na,m]} + [\mathcal{N}_{[m,dd]}]^{-1}\cdot\Delta\boldsymbol{\eta}_{\underline{w},[nd,m]}\| \\
&\le \|\boldsymbol{M}_{[m,dd]}\|\cdot\|\underline{\mathbf{u}}_{[nd,m]}\| + \|\boldsymbol{M}_{[m,da]}\|\cdot\|\underline{\mathbf{u}}_{[na,m]}\| + \|[\mathcal{N}_{[m,dd]}]^{-1}\|\cdot\|\Delta\boldsymbol{\eta}_{\underline{w},[nd,m]}\| \\
&\le \mathbf{L}_{dd}\cdot L_{dd}\cdot C\cdot\|\{\boldsymbol{y}\}_{\mathcal{M}}\| + \mathbf{L}_{dd}\cdot L_{da}\cdot\mathcal{L}_{aa}\cdot\{1 + L_{ad}\cdot C\}\cdot\|\{\boldsymbol{y}\}_{\mathcal{M}}\| + \mathbf{L}_{dd}\cdot\|\{\boldsymbol{y}\}_{\mathcal{M}}\| \\
&= \mathbf{L}_{dd}\cdot\{1 + L_{dd}\cdot C + L_{da}\cdot\mathcal{L}_{aa}\cdot\{1 + L_{ad}\cdot C\}\}\|\{\boldsymbol{y}\}_{\mathcal{M}}\|, \qquad (4.33)
\end{aligned}
$$

for $1 \le m \le \mathcal{M}$.

Utilizing (4.31), (4.32), and (4.33), we can complete (4.30) as follows:

$$
\begin{aligned}
\|\{\underline{\boldsymbol{U}}\}_{\mathcal{M}}\| &\le \max\{C,\ \mathcal{L}_{aa}\{1 + L_{ad}C\},\ \mathbf{L}_{dd}\{1 + L_{dd}C + L_{da}\mathcal{L}_{aa}\{1 + L_{ad}C\}\}\}\cdot\|\{\underline{\boldsymbol{\Gamma}}\}_{\mathcal{M}}\| \\
&=: M\cdot\|\{\underline{\boldsymbol{\Gamma}}\}_{\mathcal{M}}\|.\square \qquad (4.34)
\end{aligned}
$$

4.3 Proof of The First Hypothesis of Lemma 2.1 - Uniform Boundedness of the Inverse of the Fréchet Derivative

Now that we have completed the proof of our F.I. (4.2), we can proceed to verify (4.3) - the first hypothesis of lemma 2.1 - as follows:

Lemma 4.5

Suppose we consider nonlinear, decoupled systems of differential/algebraic equations of the form (3.1), for which $[\partial \underline{F}_a / \partial \underline{y}_a]$ is of full rank on $[x_0, x_{fin}]$. Suppose the hypothesis of lemmas 4.1, 4.2, 4.3, and 4.4 are true.

Then the Fréchet derivative, $L_h([\underline{y}]_h)$, of $\underline{G}_h([\underline{y}]_h)$ satisfies (2.12), the first hypothesis of lemma 2.1. that is, the inequality (4.3) is true.□

Proof of Lemma 4.5

The inequality (4.3) follows from our F.I. (4.2).□

4.4. Proof of the Second Hypothesis of Lemma 2.1 – Lipschitz Continuity of the Fréchet Derivative

To complete our stability analysis, we will verify the second hypothesis of lemma 2.1; that is,

Lemma 4.6

Suppose the function $\underline{F}$, in (3.1), is Lipschitz continuous with respect to $\underline{y}$ and $\underline{z}_d$ on $[x_0, x_{fin}]$. Suppose the second partial derivatives of the function $\underline{F}$ with respect to $\underline{y}$ and $\underline{z}_d$ are continuous.

Then (2.13) is true; that is, there exists a constant $K_L > 0$, such that

$$\|L_h(\underline{\mathcal{W}}_{1,h}) - L_h(\underline{\mathcal{W}}_{2,h})\| \le K_L \|\underline{\mathcal{W}}_{1,h} - \underline{\mathcal{W}}_{2,h}\|, \tag{4.35}$$

for

$$\underline{\mathcal{W}}_{j,h} \in S_{\rho_0}(\underline{\mathcal{Z}}^*_h),$$

and $j = 1, 2$.□

Proof of Lemma 4.6

The vector $\underline{\mathcal{W}}_{j,h}$ satisfies the following difference scheme

$$[hC_{[nd]}]^{-1}\{\underline{\mathbf{w}}_{j,[nd,m]} - hC_{[nd]}\underline{\mathbf{w}}'_{j,[nd,m]} - P_{[nd]}\underline{\mathbf{w}}_{j,m-1} - hR_{[nd]}\underline{\mathbf{w}}'_{j,[nd,m-1]}\}$$

$$+ \underline{\Gamma}_{j,[nd,m]} = \underline{0}_{r\cdot nd},$$

$$\underline{F}_d(x_{k+1}, \underline{w}_{j,k+1}, \underline{\mathbf{u}}_{j,k+1}) = \underline{0}_{w_j,d,k+1},$$

$$\underline{F}_a(x_{k+1}, \underline{w}_{j,k+1}) = \underline{0}_{w_j,a,k+1}, \tag{4.36}$$

where

$$\underline{\mathcal{W}}_{j,h} = [\underline{w}_{j,1}^T, \underline{\mathbf{u}}_{j,1}^T, \ldots, \underline{w}_{j,K}^T, \underline{\mathbf{u}}_{j,K}^T]^T.$$

$\underline{W}_{j,s} \approx \underline{y}(x_s)$,

$\underline{\mathbf{w}}_{j,s} \approx \underline{z}_d(x_s)$,

$\underline{\mathcal{W}}_{j,m} := [\underline{W}_{j,k+1}{}^T, \underline{W}_{j,k+2}{}^T, \ldots, \underline{W}_{j,k+r}{}^T]^T$,

$\underline{\mathcal{W}}_{j,[nd,m]} := [\underline{W}_{j,d,k+1}{}^T, \ldots, \underline{W}_{j,d,k+r}{}^T]^T$,

$\underline{\mathcal{W}}'_{j,[nd,m]} := [\underline{\mathbf{w}}_{j,k+1}{}^T, \ldots, \underline{\mathbf{w}}_{j,k+r}{}^T]^T$,

$\underline{\Gamma}_{j,[nd,m]} := [\underline{\Gamma}_{j,d,k+1}{}^T, \ldots, \underline{\Gamma}_{j,d,k+r}{}^T]^T$,

$\underline{\eta}_{\underline{w}j,m} := [\underline{\eta}_{\underline{w}j,k+1}{}^T, \ldots, \underline{\eta}_{\underline{w}j,k+r}{}^T]^T$,

$j = 1,2$,

$1 \le m \le M$,

$k = mr$,

$1 \le s \le K$,

and

$1 \le i \le r$.

Subtracting system (4.36) from system (3.4) yields the following

$$[hC_{[nd]}]^{-1}\{\underline{\mathcal{U}}_{j,[nd,m]} - hQ_{[nd]}\underline{\mathcal{U}}'_{j,[nd,m]} - P_{[nd]}\underline{\mathcal{U}}_{j,m-1} - hR_{[nd]}\underline{\mathcal{U}}'_{j,[nd,m-1]}\} = \underline{\Gamma}_{j,[nd,m]},$$

$$\underline{F}_d(x_{k+i}, \underline{y}_{k+i}, \underline{z}_{d,k+i}) - \underline{F}_d(x_{k+i}, \underline{W}_{j,k+i}, \underline{\mathbf{w}}_{j,k+i}) = \Delta\underline{\eta}_{\underline{w}j,d,k+i},$$

$$\underline{F}_a(x_{k+i}, \underline{y}_{k+i}) - \underline{F}_a(x_{k+i}, \underline{W}_{j,k+i}) = \Delta\underline{\eta}_{\underline{w}j,a,k+i}, \qquad (4.37)$$

such that

$\underline{\mathcal{U}}_{j,m} := \underline{\mathcal{Y}}_m - \underline{\mathcal{W}}_{j,m}$,

$\underline{\mathcal{U}}_{j,[nd,m]} := \underline{\mathcal{Y}}_{j,[nd,m]} - \underline{\mathcal{W}}_{j,[nd,m]}$,

$\underline{\mathcal{U}}_{j,[na,m]} := \underline{\mathcal{Y}}_{j,[na,m]} - \underline{\mathcal{W}}_{j,[na,m]}$,

$\underline{\mathcal{U}}'_{j,[nd,m]} := \underline{\mathcal{Z}}_{j,[nd,m]} - \underline{\mathcal{W}}'_{j,[nd,m]}$

$\underline{\mathscr{U}}_{j,m} := [\underline{\mathcal{U}}_{j,m}{}^T, \underline{\mathcal{U}}'_{j,[nd,m]}{}^T]^T$,

$\{\underline{\mathcal{U}}\}_{j,M} = [\underline{\mathcal{U}}_{j,1}^T, \dots, \underline{\mathcal{U}}_{j,M}^T]^T,$

$\Delta\underline{\Omega}_{\underline{w}j,m} = [\Delta\underline{\Omega}_{\underline{w}j,k+1}^T, \dots, \Delta\underline{\Omega}_{\underline{w}j,k+r}^T]^T,$

$j = 1, 2,$

$1 \le m \le M,$

$k = mr,$

and

$1 \le i \le r$

The Fréchet derivative, $L_h(\underline{w}_{j,h})$, of the mapping $G_h(\underline{w}_{j,h})$ is obtained by linearizing (4.37), that is,

$[hC_{[nd]}]^{-1}\{\underline{U}_{j,[nd,m]} - hD_{[nd]}\underline{U}'_{j,[nd,m]} - P_{[nd]}\underline{U}_{j,m-1} - hR_{[nd]}\underline{U}'_{j,[nd,m-1]}\} = \underline{\Gamma}_{j,[nd,m]},$

$[\partial\underline{F}_d(\underline{P}_{j,k+i})/\partial\underline{y}_d]\underline{u}_{j,d,k+i} + [\partial\underline{F}_d(\underline{P}_{j,k+i})/\partial\underline{y}_a]\underline{u}_{j,a,k+i} + [\partial\underline{F}_d(\underline{P}_{j,k+i})/\partial\underline{z}_d]\underline{u}'_{j,d,k+i}$

$= \Delta\underline{\Omega}_{\underline{w}j,d,k+i},$

$$[\partial\underline{F}_a(\underline{P}_{j,k+i})/\partial\underline{y}_d]\underline{u}_{j,d,k+i} + [\partial\underline{F}_a(\underline{P}_{j,k+i})/\partial\underline{y}_a]\underline{u}_{j,a,k+i} = \Delta\underline{\Omega}_{\underline{w}j,a,k+i}, \qquad (4.38)$$

such that

$\underline{P}_{j,k+i} := (t_{k+i}, \underline{Y}_{j,k+i}, \underline{w}_{j,k+i}),$

$j = 1, 2,$

$1 \le m \le M,$

$k = mr$

and

$1 \le i \le r$

Thus, the Fréchet derivative, $L_h(\underline{W}'_{j,h})$, of $G_h(\underline{W}'_{j,h})$ is defined as

$$L_h(\underline{W}'_{j,h}) := \mathrm{diag}[\mathcal{L}_1(\underline{W}'_{j,h}), \ldots, \mathcal{L}_M(\underline{W}'_{j,h})], \tag{4.39}$$

such that

$$\mathcal{L}_m(\underline{W}'_{j,h}) := \mathfrak{R}_{j,m} - \mathbf{L*}_m E^{-1}, \tag{4.40}$$

$$\mathfrak{R}_{j,m} := \left[\begin{array}{c|c} [hC_{[nd]}]^{-1}\{I_r \mathbf{X}[I_{nd}\ 0_{nd,na}]\} & [hC_{[nd]}]^{-1}[-hQ_{[nd]}] \\ \hline \mathcal{D}_{j,m} & \mathcal{D}'_{j,m} \end{array}\right]$$

$$\mathbf{L*}_m := \left[\begin{array}{c|c} [hC_{[nd]}]^{-1} P_{[nd]} & [hC_{[nd]}]^{-1}[hR_{[nd]}] \\ \hline 0_{rn} & 0_{rn,rnd} \end{array}\right],$$

$\mathcal{D}_{j,m} = \mathrm{diag}[J^{j,k+1}, \ldots, J^{j,k+r}]$,

$\mathcal{D}'_{j,m} := \mathrm{diag}[\mathbf{J}^{j,k+1}, \ldots, \mathbf{J}^{j,k+r}]$,

$$J^{j,k+1} := \begin{bmatrix} J^{j,k+i}_{dd} & J^{j,k+i}_{da} \\ J^{j,k+i}_{ad} & J^{j,k+i}_{aa} \end{bmatrix},$$

$$\mathcal{J}^{j,k+i} := \begin{bmatrix} \mathcal{J}_{dd}^{j,k+i} \\ \\ 0_{na,nd} \end{bmatrix},$$

for

$j = 1, 2,$

$1 \le m \le M,$

$k = mr,$

and

$1 \le i \le r.$

We will now look at the difference of the Fréchet derivatives of $G_h(\underline{\mathcal{W}}_{j,h})$ $(j=1,2)$ to obtain the Lipschitz condition in (4.35): that is,

$$L_h(\underline{\mathcal{W}}_{1,h})-L_h(\underline{\mathcal{W}}_{2,h}) = \mathrm{diag}[\mathcal{L}_1(\underline{\mathcal{W}}_{1,h})-\mathcal{L}_1(\underline{\mathcal{W}}_{2,h}), \ldots, \mathcal{L}_{M}(\underline{\mathcal{W}}_{1,h})-\mathcal{L}_{M}(\underline{\mathcal{W}}_{2,h})], \qquad (4.41)$$

where

$$\mathcal{L}_m(\underline{\mathcal{W}}_{1,h}) - \mathcal{L}_m(\underline{\mathcal{W}}_{2,h}) := \{R_{1,m} - L_{*m}E^{-1}\} - \{R_{2,m} - L_{*m}E^{-1}\}$$

$$:= \mathcal{R}_{1,m} - \mathcal{R}_{2,m}$$

$$:= \left[\begin{array}{c|c} 0_{rn} & 0_{r\,nd} \\ \hline \mathcal{D}_{1,m}-\mathcal{D}_{2,m} & \mathcal{D}'_{1,m}-\mathcal{D}'_{2,m} \end{array}\right], \qquad (4.42)$$

$\mathcal{D}_{1,m} - \mathcal{D}_{2,m} := \mathrm{diag}[J^{1,k+1} - J^{2,k+1}, \dots, J^{1,k+r} - J^{2,k+r}],$

$\mathcal{D}'_{1,m} - \mathcal{D}'_{2,m} := \mathrm{diag}[\mathfrak{J}^{1,k+1} - \mathfrak{J}^{2,k+1}, \dots, \mathfrak{J}^{1,k+r} - \mathfrak{J}^{2,k+r}],$

and

$1 \le m \le \mathcal{M}.$

Via the hypothesis that the second partial derivatives of $\underline{F}$ with respect to $\underline{y}$ and $\underline{z}_d$ are continuous, we can infer that the first partial derivatives of $\underline{F}$ with respect to $\underline{y}$ and $\underline{z}_d$ are Lipschitz continuous with respect to $\underline{y}$ and $\underline{z}_d$ on $[x_0, x_{fin}]$. Thus there exists positive constants K_s $(s=1,2)$, such that

$$\|J^{1,k+i} - J^{2,k+i}\| \le K_1 \|\underline{w}_{1,k+i} - \underline{w}_{2,k+i}\|, \tag{4.43a}$$

and

$$\|\mathfrak{J}^{1,k+i} - \mathfrak{J}^{2,k+i}\| \le K_2 \|\underline{\mathfrak{w}}_{1,k+i} - \underline{\mathfrak{w}}_{2,k+i}\|, \tag{4.43b}$$

for

$1 \le i \le r,$

$k = mr,$

and

$1 \le m \le \mathcal{M}.$

Utilizing (4.41), (4.42) and (4.43), we obtain.

$\|L_h(\underline{\mathcal{W}}_{1,h}) - L_h(\underline{\mathcal{W}}_{2,h})\| \le \max\{\|\mathcal{L}_m(\underline{\mathcal{W}}_{1,h}) - \mathcal{L}_m(\underline{\mathcal{W}}_{2,h})\|: 1 \le m \le \mathcal{M}\}$

$\le \max\{\max\{\|\mathcal{D}_{1,m} - \mathcal{D}_{2,m}\|, \|\mathcal{D}'_{1,m} - \mathcal{D}'_{2,m}\|: 1 \le m \le \mathcal{M}\}\}$

$\le \max\{\max\{\|J^{1,k+i} - J^{2,k+i}\|, \|\mathfrak{J}^{1,k+i} - \mathfrak{J}^{2,k+i}\|: 1 \le i \le r, k = mr, 1 \le m \le \mathcal{M}\}\}$

$\le \max\{\max\{K_1\|\underline{w}_{1,k+i} - \underline{w}_{2,k+i}\|, K_2\|\underline{\mathfrak{w}}_{1,k+i} - \underline{\mathfrak{w}}_{2,k+i}\|: 1 \le i \le r, k = mr, 1 \le m \le \mathcal{M}\}\}$

$\le \max\{K_1, K_2\} \|\underline{\mathcal{W}}_{1,h} - \underline{\mathcal{W}}_{2,h}\|$

$=: K_L \|\underline{\mathcal{W}}_{1,h} - \underline{\mathcal{W}}_{2,h}\|. \square$

We have now set things up to invoke lemma 2.1 from [Kel75]. To arrive at this point we stated and proved various lemmas. These results are summarized as follows.

Theorem 4.1

Consider nonlinear, decoupled systems of differential/algebraic equations of the form (3.1). Let

$\underline{y} := [\underline{y}^T, \underline{z}_d^T]^T$,

be a unique local analytical solution to (3.1). Suppose the function $\underline{F}$ in (3.1) is such that $[\partial \underline{F}_a / \partial \underline{y}_a]$ and $[\partial \underline{F}_d / \partial \underline{z}_d]$ are of full rank on $[x_0, x_{fin}]$. Suppose the hypothesis of lemmas 4.1, 4.2, 4.3, 4.4, 4.5, and 4.6 are true.

Then our FSFF block implicit one-step methods (1.9) applied to nonlinear, decoupled index one systems of differential/algebraic equations of the form (3.1) are stable. ☐

5. FSFF Convergence Analysis for Smooth Index One Systems of Differential/Algebraic Equations

We will now utilize the results obtained in the previous subsections to prove convergence of our FSFF block implicit one-step methods (1.9) applied to smooth, nonlinear, decoupled index one systems of differential/algebraic equations of the form (3.1). Essentially, we have set things up to invoke Theorem 2.2 from [4] to obtain the classical convergence result – which was stated in theorem 2.1; that is,

Theorem 5.1

Consider nonlinear, decoupled index one systems of differential/algebraic equations of the form (3.1), for which the first and second derivatives exist and are continuous. Let

$$\underline{y} := [\underline{y}^T, \underline{z}_d^T]^T$$

be a unique local analytical solution to (3.1). Suppose the hypothesis of theorems 3.1 and 4.1 are satisfied

Then for some $\rho_0 > 0$, and

$$h := (x_{fin} - x_0)/K,$$

the problem

$$G_h(v_h) = 0$$

has a unique solution $v_h = w_h \in S_{\rho_0}([\underline{y}]_h)$, which converges to the analytical solution of

$$G(\underline{y}) = 0$$

that is

$$\lim_{h \to 0} \|[\underline{y}]_h - w_h\| \to 0$$

Also, the order of this convergence is

$$p = r+1$$

that is, there exists a positive constant M_{con}, such that

$$\|[\underline{y}]_h - w_h\| \leq M_{con} h^p . \square$$

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